Periodic structures separated by highly conductive thin layers

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Isabelle Gruais * **and Dan Poliševski** **

Abstract. We derive a model for the conduction in an ε-periodic structure containing highly conductive thin layers. The case of plane thin layers is first considered. It is shown that the resulting model displays an increased conductivity along the directions of the layers planes. The more involved case of tubular layers yields a similar result with an increase of the conductivity along the direction of the tubes, while the presence of highly conductive thin layers confined between spheres of ε-order radii does not increase the macroscopic conductivity in any direction. It seems that the presence of ε-periodic highly conductive thin layers determines an increase of the macroscopic conductivity in a certain direction only if these layers cover entirely segment lines of unity order having this direction.

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1 Introduction

The study of layered materials is one main achievement of homogenization theory, beside Darcy's law in fluid mechanics and the modelling of composite materials in elasticity. The foundations of layered materials were laid down by Murat and Tartar in their pioneering work [9]. It states that given some characteristic coefficients a_ε, under some additional assumptions on the inverse $1/a_\varepsilon$, the reduced model can be explicited. In [6], the theory still works in the framework of weaker topologies. The case of BV-functions and sequences of measures is worked out in [5]. The engineering point of view favors geometric considerations and industrial achievements. In that respect, measures provide a more realistic tool when they are confined to the description of the critical part of a system as in [1], [2]. Later, the control-zone method introduced in [3], [4] and designed for Sobolev spaces, proved efficient in the modelling of fine substructures where small particles of high density influence the behavior of the global problem in spite of their vanishing volume. The asymptotic treatment reveals the apparent paradox between an obviously disappearing element and its everlasting action on their environment.

The paper is organized as follows. The problem of plane thin layers is considered in Section 2. Subsection 2.1 is devoted to the main notations and to the description of the initial problem. The functional framework is introduced through the space W_1 in (11)–(54) and yields the existence and unicity of the solution. Subsection 2.2 introduces the main tools of a control-zone (18) characterized by two layers widths r_ε and R_ε. Then, the problem reduces to studying the action of both local operators G_{R_ε} and G_{r_ε} on the solution u_ε of the initial problem. The homogenization procedure is described in Subsection 2.3. The main theorem 3.21 emphasizes the influence of the geometry of the fissures on the global behavior of the mixture in the form of an enforcing multiplicative coefficient in the plane direction. Interestingly, we observe that unlike the case of a geometry involving a capacity criterium, there is no discriminant parameter and the result holds however very small the thickness r_ε of a layer is in comparison to the size of the distribution period.

The more involved case of tubular layers is studied in Section 3. The arguments follow the same lines with an exchange in the respective parts of the dimensions, namely the plane dimension containing the periodic distribution of the tubes where the homogenization actually takes place and the dimension corresponding to the direction of the tubes. This is explained in Subsection 3.1 which is the analogue of Subsection 2.1. The control tools are presented in Subsection 3.2 where the new definition of local operators involves the mean value upon both interior and exterior boundaries of the tubular layers. From a practical point of view, the mere exchange in the respective parts of the plane and longitudinal coordinates corresponds to an improved theoretical material. In the main Theorem of Subsection 3.3, the respective parts of the thin tubular fissures and the surrounding mixture is emphasized through the introduction of a scaled conductivity outside the fissures. It eventually shows that the global conductivity is increased in the direction of the fissures and that this increase adds to the global conductivity that is classically derived in the presence of a periodic in-plane distribution.

The same arguments apply to the homogenization of thin layers confined between spheres of ε-order radii, but with no increase of the macroscopic conductivity, in any direction. Indeed, the macroscopic conductivity takes into account both components of the original system only if the thin layers cover segment lines of unity order having a certain direction. Eventually, the increase of the macroscopic conductivity shows up along this distinguished directions of the thin layers.

2 The case of separating thin plane layers

2.1 The conduction problem

We consider $\Omega = I \times D$ with $I = (0,1)$, $D \subseteq \mathbf{R}^N$, $N \geq 1$ a bounded Lipschitz domain occupied by a mixture of two different materials, one of them forming the ambiental connected phase and the other being concentrated in a periodical distribution of plane thin layers. Let us denote

$$I := \left(-\frac{1}{2}, +\frac{1}{2}\right). \tag{1}$$

$$I_\varepsilon^k := \varepsilon k + \varepsilon I, \quad k \in \mathbf{Z}_\varepsilon := \{\ell \in \mathrm{N},\ 0 < \varepsilon\ell < 1\}. \tag{2}$$

The distribution of fissures is defined by the following reunion

$$T_\varepsilon := \cup_{k\in\mathbf{Z}_\varepsilon} I_{r_\varepsilon}^k \times D, \tag{3}$$

where $0 < r_\varepsilon << \varepsilon$ and $I_{r_\varepsilon}^k := \varepsilon k + 2r_\varepsilon I$. Obviously,

$$|T_\varepsilon| = 2r_\varepsilon \mathrm{card}\, \mathbf{Z}_\varepsilon |D| \to 0 \quad \text{as} \quad \varepsilon \to 0. \tag{4}$$

We also use the following notation

$$\Omega_\varepsilon = \left(\frac{\varepsilon}{2}, 1 - \frac{\varepsilon}{2}\right) \times D, \quad \partial_1\Omega := \{0,1\} \times D, \quad \partial_D\Omega := (0,1) \times \partial D. \tag{5}$$

We consider the problem which governs the conduction process throughout our binary mixture. Denoting by $a_\varepsilon > 0$ the relative conductivity of the thin layers, then, its non-dimensional form is the following:

To find u_ε solution of

$$-\mathrm{div}(A_\varepsilon \nabla u_\varepsilon) = f_\varepsilon \quad \text{in} \quad \Omega \tag{6}$$

$$u_\varepsilon = 0 \quad \text{on} \quad \partial_1 D \tag{7}$$

$$\frac{\partial u_\varepsilon}{\partial \nu} = 0, \quad \nu = (0, \nu') \quad \text{on} \quad \partial_D \Omega \tag{8}$$

$$u_\varepsilon = 0 \quad \text{on} \quad \partial\Omega \tag{9}$$

where ν' is the normal on ∂D in the outward direction, $f_\varepsilon \in L^2(\Omega)$ and

$$A_\varepsilon(x) = \begin{cases} 1 & \text{if} \quad x \in \Omega \setminus T_\varepsilon \\ a_\varepsilon & \text{if} \quad x \in T_\varepsilon. \end{cases} \tag{10}$$

Let W_1 be the Hilbert space

$$W_1 := \{v \in H^1(\Omega), \quad v = 0 \quad \text{on} \quad \partial_1 D\} \tag{11}$$

endowed with the scalar product

$$(u, v)_{W_1} := (\nabla u, \nabla v)_\Omega. \tag{12}$$

Now, we can present the variational formulation of the problem (6)–(10).

To find $u_\varepsilon \in W_1$ satisfying the following equation

$$\int_{\Omega\setminus T_\varepsilon} \nabla u_\varepsilon \nabla v + a_\varepsilon \int_{T_\varepsilon} \nabla u_\varepsilon \nabla v = \langle f_\varepsilon, v\rangle, \quad \forall w \in W_1 \tag{13}$$

where $\langle\cdot,\cdot\rangle$ denotes the duality product between W_1 and W_1'.

Theorem 2.1 *Under the above hypotheses and notations, problem (13) has a unique solution.*

In the following we consider that the conductivity of the layers is much higher than that of the surrounding phase. The specific feature of our mixture is given by the following relation which describes the fact that the conductivity of the thin layers is balanced by their vanishing volume:

$$\lim_{\varepsilon\to 0} a_\varepsilon |T_\varepsilon| = \eta > 0. \tag{14}$$

As for the data, we assume that there exists $f \in W_1'$ such that

$$f_\varepsilon \rightharpoonup f \quad \text{in} \quad W_1'. \tag{15}$$

Also, we denote

$$⨍_D \cdot\, dx = \frac{1}{|D|}\int_D \cdot dx.$$

Proposition 2.2 *We have*

$$(u_\varepsilon)_\varepsilon \quad \textit{is bounded in } W_1. \tag{16}$$

Moreover, there exists $C > 0$, independent of ε, such that

$$⨍_{T_\varepsilon} |\nabla u_\varepsilon|^2 \leq C. \tag{17}$$

Proof. Substituting $w = u_\varepsilon$ in the variational problem (13), and taking into account that

$$|v|_{W_1} \leq C|\nabla v|_{L^2(\Omega)}, \quad \forall v \in W_1,$$

we get:

$$C\int_\Omega |\nabla u_\varepsilon|^2 \leq \int_{T_\varepsilon} |\nabla u_\varepsilon|^2 + a_\varepsilon \int_{\Omega\setminus T_\varepsilon} |\nabla u_\varepsilon|^2 \leq C|f_\varepsilon|_{W_1'}|\nabla u_\varepsilon|_{L^2(\Omega)}$$

There results:

$$|\nabla u_\varepsilon|_{L^2(\Omega)} \leq C$$

and the proof is completed. ∎

2.2 Specific tools

The set of control-sequences is defined by

$$\mathcal{R} = \{(R_\varepsilon)_{\varepsilon>0}, \quad r_\varepsilon << R_\varepsilon << \varepsilon\} \tag{18}$$

that is $(R_\varepsilon)_{\varepsilon>0} \in \mathcal{R}$ iff

$$\lim_{\varepsilon\to 0} \frac{r_\varepsilon}{R_\varepsilon} = \lim_{\varepsilon\to 0} \frac{R_\varepsilon}{\varepsilon} = 0. \tag{19}$$

For any $(R_\varepsilon)_\varepsilon \in \mathcal{R}$, we denote the domain confined between the layers of widths r_ε and R_ε as the union of all subsets constructed from

$$\mathcal{C}^k_{R_\varepsilon} := I^k_{R_\varepsilon} \setminus \overline{I^k_{r_\varepsilon}}, \quad \text{where} \quad I^k_{R_\varepsilon} := \varepsilon k + 2R_\varepsilon I$$

namely:

$$\mathcal{C}_\varepsilon := (\cup_{k\in \mathbf{Z}_\varepsilon} \mathcal{C}^k_{R_\varepsilon}) \times D.$$

Then:

$$|\mathcal{C}_\varepsilon| = \mathrm{card}\mathbf{Z}_\varepsilon (R_\varepsilon - r_\varepsilon)|D| \to 0 \quad \text{as} \quad \varepsilon \to 0. \tag{20}$$

Definition 2.3 *For any $(R_\varepsilon)_\varepsilon \in \mathcal{R}$, we define $w_\varepsilon \in W_1$ by*

$$w_\varepsilon(x_1, x') := \begin{cases} 1 - \dfrac{r_\varepsilon}{R_\varepsilon} & \text{if} \quad (x_1, x') \in \overline{T}_\varepsilon, \\ \\ 1 - \dfrac{1}{R_\varepsilon}\,|x_1 - \varepsilon k| & \text{if} \quad (x_1, x') \in \mathcal{C}^k_{R_\varepsilon} \times D, \quad k \in \mathbf{Z}_\varepsilon, \\ \\ 0 & \text{if} \quad (x_1, x') \in \Omega \setminus (\overline{T}_\varepsilon \cup \mathcal{C}_\varepsilon). \end{cases} \tag{21}$$

We remark here the following properties of w_ε:

$$w_\varepsilon \in W_1 \tag{22}$$

$$0 \leq w_\varepsilon(x) < 1, \quad \forall x \in \Omega \tag{23}$$

$$|w_\varepsilon|_{L^2(\Omega)} \leq |T_\varepsilon \cup \mathcal{C}_\varepsilon|^{1/2} \leq C\sqrt{\frac{R_\varepsilon}{\varepsilon}} \to 0. \tag{24}$$

Definition 2.4 *For any $u \in W_1$ and any $(b_\varepsilon)_{n\in \mathrm{N}}$ with $0 < b_\varepsilon < \dfrac{\varepsilon}{2}$, we define $G^k_{b_\varepsilon}(u) \in H^{1/2}(D)$ and $G_{b_\varepsilon}(u) \in L^2(\Omega)$ by*

$$G^k_{b_\varepsilon}(u)(x') = \frac{1}{2}\left(u\left(\varepsilon k - b_\varepsilon, x'\right) + u\left(\varepsilon k + b_\varepsilon, x'\right)\right)$$

$$G_{b_\varepsilon}(u)(x_1, x') = \sum_{k\in \mathbf{Z}_\varepsilon} G^k_{b_\varepsilon}(u)(x') 1_{I^k_\varepsilon}(x_1), \quad (x_1, x') \in \Omega.$$

Proposition 2.5 *For any $u \in W_1$, we have:*

$$\fint_{T_\varepsilon} |G_{r_\varepsilon}(u)|^2 = \frac{1}{\varepsilon \mathrm{card}\mathbf{Z}_\varepsilon} \fint_\Omega |G_{r_\varepsilon}(u)|^2. \tag{25}$$

Proof. We have

$$\int_\Omega |G_{r_\varepsilon}(u)|^2 = \sum_{k\in \mathbf{Z}_\varepsilon} \int_{I\times D} |G^k_{r_\varepsilon}(u)(x')|^2 1_{I^k_\varepsilon} = \varepsilon \sum_{k\in \mathbf{Z}_\varepsilon} \int_D |G^k_{r_\varepsilon}(u)|^2.$$

Moreover:

$$\fint_{T_\varepsilon} |G_{r_\varepsilon}(u)|^2 = \frac{1}{2\mathrm{card}\mathbf{Z}_\varepsilon r_\varepsilon |D|} \sum_{k\in \mathbf{Z}_\varepsilon} \int_{I^k_{r_\varepsilon}\times D} |G_{r_\varepsilon}(u)|^2(x') 1_{I^k_\varepsilon}(x_1) =$$

$$= \frac{1}{\mathrm{card}\mathbf{Z}_\varepsilon |D|} \sum_{k\in\mathbf{Z}_\varepsilon} \int_D |G^k_{r_\varepsilon}(u)|^2.$$

This achieves the proof. ■

Proposition 2.6 *For any $u \in W_1$, there holds:*

$$\int_{I^k_\varepsilon \times D} |u - G^k_{b_\varepsilon}(u)|^2 \leq \frac{\varepsilon^2}{2} \int_{I^k_\varepsilon \times D} \left|\frac{\partial u}{\partial x_1}\right|^2 \tag{26}$$

Proof. Consider the quantity

$$J^k_\varepsilon = \int_{I^k_\varepsilon \times D} |u - G^k_{b_\varepsilon}(u)|^2.$$

We have

$$J^k_\varepsilon = \int_{I^k_\varepsilon} dx_1 \int_D \left| u(x_1, x') - \frac{1}{2}\left(u\left(\varepsilon k - b_\varepsilon, x'\right) + u\left(\varepsilon k + b_\varepsilon, x'\right)\right)\right|^2 dx' =$$

$$= \frac{1}{4} \int_{I^k_\varepsilon} dx_1 \int_D \left| \int_{\varepsilon k - b_\varepsilon}^{x_1} \frac{\partial u}{\partial x_1}(t, x')dt - \int_{x_1}^{\varepsilon k + b_\varepsilon} \frac{\partial u}{\partial x_1}(t, x')dt \right|^2 dx' \leq$$

$$\leq \frac{1}{2} \int_{I^k_\varepsilon} dx_1 \int_D \left(\left| \int_{\varepsilon k - b_\varepsilon}^{x_1} \frac{\partial u}{\partial x_1}(t, x')dt \right|^2 + \left| \int_{x_1}^{\varepsilon k + b_\varepsilon} \frac{\partial u}{\partial x_1}(t, x')dt \right|^2 \right) dx' \leq$$

$$\leq \frac{1}{2}\int_{I^k_\varepsilon} dx_1 \int_D \left[(x_1 - \varepsilon k + b_\varepsilon) \int_{\varepsilon k - b_\varepsilon}^{x_1} \left|\frac{\partial u}{\partial x_1}(t, x')\right|^2 dt + (\varepsilon k + b_\varepsilon - x_1) \int_{x_1}^{\varepsilon k + b_\varepsilon} \left|\frac{\partial u}{\partial x_1}(t, x')\right|^2 dt \right] dx'.$$

There results

$$J^k_\varepsilon \leq \frac{\varepsilon}{2} \int_{I^k_\varepsilon} dx_1 \underbrace{\int_D dx' \int_{\varepsilon k - \frac{\varepsilon}{2}}^{\varepsilon k + \frac{\varepsilon}{2}} \left|\frac{\partial u}{\partial x_1}(t, x')\right|^2 dt}_{\text{independent of } x_1} = \frac{\varepsilon^2}{2} \int_{I^k_\varepsilon \times D} \left|\frac{\partial u}{\partial x_1}\right|^2$$

■

Proposition 2.7 *For any $u \in W_1$, there holds*

$$\int_{I^k_{b_\varepsilon} \times D} |u - G^k_{b_\varepsilon}(u)|^2 \leq 2b_\varepsilon^2 \int_{I^k_{b_\varepsilon} \times D} \left|\frac{\partial u}{\partial x_1}\right|^2 \tag{27}$$

Proof. We have

$$\int_{I^k_{b_\varepsilon} \times D} |u - G^k_{b_\varepsilon}(u)|^2 = \int_{I^k_{b_\varepsilon}} dx_1 \int_D \left| u(x_1, x') - \frac{1}{2}\left(u\left(\varepsilon k - b_\varepsilon, x'\right) + u\left(\varepsilon k + b_\varepsilon, x'\right)\right)\right|^2 dx' dx_1 \leq$$

$$\leq \frac{1}{2} \int_{I^k_{b_\varepsilon}} \int_D \left(\left| \int_{\varepsilon k - b_\varepsilon}^{x_1} \frac{\partial u}{\partial x_1}(t, x')dt \right|^2 + \left| \int_{x_1}^{\varepsilon k + b_\varepsilon} \frac{\partial u}{\partial x_1}(t, x')dt \right|^2 \right) dx' \leq 2b_\varepsilon^2 \int_{I^k_{b_\varepsilon} \times D} \left|\frac{\partial u}{\partial x_1}\right|^2$$

■

Proposition 2.8 *For any $u \in W_1$, we have:*

$$|G_{r_\varepsilon}(u) - G_{R_\varepsilon}(u)|_{L^2(\Omega)} \leq (\varepsilon R_\varepsilon)^{1/2} \left|\frac{\partial u}{\partial x_1}\right|_{L^2(\mathcal{C}_\varepsilon)} \tag{28}$$

Proof. For each k, $k \in \mathbf{Z}_\varepsilon$, there holds:

$$\int_D \left|G^k_{r_\varepsilon}(u)(x') - G^k_{R_\varepsilon}(u)(x')\right|^2 dx' \leq (R_\varepsilon - r_\varepsilon) \int_{\mathcal{C}^k_{R_\varepsilon} \times D} \left|\frac{\partial u}{\partial x_1}\right|^2.$$

Indeed, let $k \in \mathbf{Z}_\varepsilon$. We have

$$\int_D \left|G^k_{r_\varepsilon}(u)(x') - G^k_{R_\varepsilon}(u)(x')\right|^2 dx' \leq \frac{1}{2}\int_D \left(\left|\int_{\varepsilon k - R_\varepsilon}^{\varepsilon k - r_\varepsilon} \frac{\partial u}{\partial x_1}(t, x')dt\right|^2 + \left|\int_{\varepsilon k + r_\varepsilon}^{\varepsilon k + R_\varepsilon} \frac{\partial u}{\partial x_1}(t, x')dt\right|^2\right) dx' \leq$$

$$\leq (R_\varepsilon - r_\varepsilon) \int_{\mathcal{C}^k_{R_\varepsilon} \times D} \left|\frac{\partial u}{\partial x_1}\right|^2.$$

To conclude about (28), we write:

$$|G_{r_\varepsilon}(u) - G_{R_\varepsilon}(u)|^2_{L^2(\Omega)} = \sum_{k \in \mathbf{Z}_\varepsilon} \int_{I^k_\varepsilon \times D} \left|G^k_{r_\varepsilon}(u_\varepsilon) - G^k_{R_\varepsilon}(u_\varepsilon)\right|^2 =$$

$$= \varepsilon \sum_{k \in \mathbf{Z}_\varepsilon} \int_D \left|G^k_{r_\varepsilon}(u)(x') - G^k_{R_\varepsilon}(u)(x')\right|^2 dx' \leq \varepsilon(R_\varepsilon - r_\varepsilon) \sum_{k \in \mathbf{Z}_\varepsilon} \int_{\mathcal{C}^k_{R_\varepsilon} \times D} \left|\frac{\partial u}{\partial x_1}\right|^2 \leq \varepsilon R_\varepsilon \left|\frac{\partial u}{\partial x_1}\right|^2_{L^2(\mathcal{C}_\varepsilon)}$$

■

Proposition 2.9 *The following estimate holds true:*

$$\fint_{T_\varepsilon} |u|^2 \leq C|\nabla u|^2_{L^2(\Omega)}, \quad \forall u \in W_1 \tag{29}$$

for some constant $C > 0$ independent of ε.

Proof. From (25) and (27), we have

$$\fint_{T_\varepsilon} |u|^2 \leq 2\fint_{T_\varepsilon} |u - G_{r_\varepsilon}(u)|^2 + 2\fint_{T_\varepsilon} |G_{r_\varepsilon}(u)|^2 \leq$$

$$\leq C\varepsilon r_\varepsilon \left|\frac{\partial u}{\partial x_1}\right|^2_{L^2(T_\varepsilon)} + \frac{1}{\varepsilon \mathrm{card}\mathbf{Z}_\varepsilon} \fint_\Omega |G_{r_\varepsilon}(u)|^2.$$

Moreover, using (28) and the Poincaré-Wirtinger inequality in Ω, we get

$$|G_{r_\varepsilon}(u)|_{L^2(\Omega)} \leq |G_{r_\varepsilon}(u) - G_{R_\varepsilon}(u)|_{L^2(\Omega)} + |G_{R_\varepsilon}(u) - u|_{L^2(\Omega)} + |u|_{L^2(\Omega)} \leq$$

$$\leq (\varepsilon R_\varepsilon)^{1/2} \left|\frac{\partial u}{\partial x_1}\right|_{L^2(\mathcal{C}_\varepsilon)} + \varepsilon \left|\frac{\partial u}{\partial x_1}\right|_{L^2(\Omega)} + C|\nabla u|_{L^2(\Omega)}.$$

■

Corollary 2.10 *The following estimate holds true:*

$$\fint_{T_\varepsilon} |u_\varepsilon|^2 \leq C \tag{30}$$

for some constant $C > 0$ independent of ε.

Definition 2.11 *For every $\varphi \in C(\overline{\Omega}) \cap W_1$, define*

$$M_{r_\varepsilon}(\varphi) = \sum_{k \in \mathbf{Z}_\varepsilon} \left(\fint_{I_\varepsilon^k} \varphi(t, x')dt \right) 1_{I_{r_\varepsilon}^k}(x_1)$$

Proposition 2.12 *There holds*

$$\lim_{\varepsilon \to 0} \fint_{T_\varepsilon} |\varphi - M_{r_\varepsilon}(\varphi)|^2 = 0, \quad \forall \varphi \in C(\overline{\Omega}) \cap W_1. \tag{31}$$

Proof. Let $\varphi \in C(\overline{\Omega}) \cap W_1$ and let $\delta > 0$. The uniform continuity of φ yields the existence of some $\varepsilon_0 > 0$ such that:

$$\int_{I_{r_\varepsilon}^k} |\varphi(x_1, x') - \fint_{I_\varepsilon^k} \varphi(t, x')dt|^2 < \delta, \quad \forall x' \in D, \quad \forall k \in \mathbf{Z}_\varepsilon, \quad \forall \varepsilon \in (0, \varepsilon_0).$$

There results

$$\fint_{T_\varepsilon} |\varphi - M_{r_\varepsilon}(\varphi)| = \frac{1}{|T_\varepsilon|} \sum_{k \in \mathbf{Z}_\varepsilon} \int_{I_{r_\varepsilon}^k \times D} |\varphi(x_1, x') - \fint_{I_\varepsilon^k} \varphi(t, x')dt| \leq \delta, \ \forall \varepsilon \in (0, \varepsilon_0).$$

∎

Proposition 2.13 *There holds*

$$\fint_{T_\varepsilon} G_{r_\varepsilon}(u_\varepsilon) M_{r_\varepsilon}(\varphi) = \frac{1}{\varepsilon \text{card} \mathbf{Z}_\varepsilon} \fint_\Omega G_{r_\varepsilon}(u_\varepsilon)\varphi, \quad \forall \varphi \in C(\overline{\Omega}) \cap W_1. \tag{32}$$

Proof. Let $\varphi \in C(\overline{\Omega}) \cap W_1$. We have:

$$\fint_{T_\varepsilon} G_{r_\varepsilon}(u_\varepsilon) M_{r_\varepsilon}(\varphi) = \frac{1}{|T_\varepsilon|} \sum_{k \in \mathbf{Z}_\varepsilon} \int_{I_{r_\varepsilon}^k \times D} G_{r_\varepsilon}^k(u_\varepsilon)(x') \left(\fint_{I_\varepsilon^k} \varphi(t, x')dt \right) dx =$$

$$= \frac{2r_\varepsilon}{2\varepsilon \text{card} \mathbf{Z}_\varepsilon \ r_\varepsilon |D|} \sum_{k \in \mathbf{Z}_\varepsilon} \int_{I_\varepsilon^k \times D} G_{r_\varepsilon}^k(u_\varepsilon)(x')\varphi(t, x')dx'dt =$$

$$= \frac{1}{\varepsilon \text{card} \mathbf{Z}_\varepsilon |D|} \int_\Omega G_{r_\varepsilon}(u_\varepsilon)\varphi.$$

∎

2.3 The homogenization result

A preliminary result is the following:

Proposition 2.14 *There exists $u \in W_1$ such that, on some subsequence,*

$$u_\varepsilon \rightharpoonup u \quad in \quad H^1(\Omega). \tag{33}$$

$$G_{R_\varepsilon}(u_\varepsilon) \to u \quad in \quad L^2(\Omega) \tag{34}$$

$$G_{r_\varepsilon}(u_\varepsilon) \to u \quad in \quad L^2(\Omega) \tag{35}$$

Proof. From (16), we get, on some subsequence, the convergence (33). Moreover, from (26), we have:

$$|u_\varepsilon - G_{R_\varepsilon}(u_\varepsilon)|^2_{L^2(\Omega)} = \sum_{k\in\mathbf{Z}_\varepsilon} \int_{I^k_\varepsilon \times D} |u_\varepsilon - G^k_{R_\varepsilon}(u_\varepsilon)|^2 \leq \varepsilon^2 \left|\frac{\partial u_\varepsilon}{\partial x_1}\right|^2_{L^2(\Omega)} \leq C\varepsilon^2.$$

and the proof of (34) is complete.
In order to prove (35), we recall (28), which yields:

$$|G_{r_\varepsilon}(u_\varepsilon) - G_{R_\varepsilon}(u_\varepsilon)|_{L^2(\Omega)} \leq C\,(\varepsilon R_\varepsilon)^{1/2} \to 0.$$

and the conclusion follows from (34).

$\blacksquare$

Proposition 2.15 *For any $\varphi \in W_1$, the following convergences hold true on some subsequence:*

$$⨍_{T_\varepsilon} u_\varepsilon \varphi \to ⨍_\Omega u\varphi, \tag{36}$$

$$⨍_{T_\varepsilon} \frac{\partial u_\varepsilon}{\partial x_i}\varphi \to ⨍_\Omega \frac{\partial u}{\partial x_i}\varphi, \quad i = 2, 3, \cdots \tag{37}$$

where u was introduced by (33).

Proof. Let $\varphi \in C(\Omega) \cap W_1$. We have

$$⨍_{T_\varepsilon} u_\varepsilon\varphi = ⨍_{T_\varepsilon} (u_\varepsilon - G_{r_\varepsilon}(u_\varepsilon))\varphi + ⨍_{T_\varepsilon} G_{r_\varepsilon}(u_\varepsilon)(\varphi - M_{r_\varepsilon}(\varphi)) + ⨍_{T_\varepsilon} G_{r_\varepsilon}(u_\varepsilon) M_{r_\varepsilon}(\varphi).$$

Taking into account (27) and (31), we have:

$$\left|⨍_{T_\varepsilon} (u_\varepsilon - G_{r_\varepsilon}(u_\varepsilon))\varphi\right| \leq C r_\varepsilon \left(⨍_{T_\varepsilon} \left|\frac{\partial u_\varepsilon}{\partial x_1}\right|^2\right)^{1/2} |\varphi|_{L^\infty(\Omega)} \leq C r_\varepsilon \to 0$$

$$\left|⨍_{T_\varepsilon} G_{r_\varepsilon}(u_\varepsilon)(\varphi - M_{r_\varepsilon}(\varphi))\right| \leq |G_{r_\varepsilon}(u_\varepsilon)|_{L^2(\Omega)} \left(⨍_{T_\varepsilon} |\varphi - M_{r_\varepsilon}(\varphi)|^2\right)^{1/2} \to 0.$$

Then, using (32), we get

$$\lim_{\varepsilon\to 0} ⨍_{T_\varepsilon} u_\varepsilon\varphi = \lim_{\varepsilon\to 0} \frac{1}{\varepsilon \mathrm{card}\mathbf{Z}_\varepsilon} ⨍_\Omega G_{r_\varepsilon}(u_\varepsilon)\varphi = ⨍_\Omega u\varphi.$$

In order to conclude (36), we notice that $C(\Omega)\cap W_1$ is dense in W_1 and that the following estimate proves the continuity of the mapping $\varphi \mapsto \fint_{T_\varepsilon} u_\varepsilon \varphi$:

$$\left|\fint_{T_\varepsilon} u_\varepsilon\varphi\right| \leq \left(\fint_{T_\varepsilon} |u_\varepsilon|^2\right)^{1/2}\left(\fint_{T_\varepsilon} |\varphi|^2\right)^{1/2} \leq C|\nabla\varphi|_{L^2(\Omega)}.$$

For $i = 2, 3, \cdots$, and for $\varphi \in \mathcal{D}(\Omega)$, we have

$$\int_{T_\varepsilon} \frac{\partial}{\partial x_i}(u_\varepsilon\varphi) = \int_{\partial T_\varepsilon} u_\varepsilon\varphi\nu_i = \sum_{k\in\mathbf{Z}_\varepsilon}\int_{\partial\mathcal{C}^k_{R_\varepsilon}\times D} u_\varepsilon\varphi\nu_i = 0$$

as $\varphi = 0$ on $\partial_D\Omega$ and $\nu_i = 0$ on $\partial_1\Omega$. There results:

$$\fint_{T_\varepsilon} \frac{\partial u_\varepsilon}{\partial x_i}\varphi + \fint_{T_\varepsilon} u_\varepsilon\frac{\partial\varphi}{\partial x_i} = 0. \tag{38}$$

As a consequence of the estimate (30), there exist v_2 and $v_3 \in H^{-1}(\Omega)$ such that:

$$\lim_{\varepsilon\to 0}\fint_{T_\varepsilon} \frac{\partial u_\varepsilon}{\partial x_i}\varphi = \langle v_i, \varphi\rangle.$$

Then, passing to the limit as $\varepsilon \to 0$, we get

$$\langle v_i, \varphi\rangle + \fint_\Omega u\frac{\partial\varphi}{\partial x_i} = 0. \tag{39}$$

As (39) holds for every $\varphi \in \mathcal{D}(\Omega)$, we infer that $v_i = \dfrac{1}{|\Omega|}\dfrac{\partial u}{\partial x_i} \in L^2(\Omega)$ and the proof is completed. ■

Definition 2.16 *For any $\varphi \in W_1 \cap C^1(\overline{\Omega})$, we define $\hat{\varphi}_\varepsilon \in L^\infty(\Omega)$ by*

$$\hat{\varphi}_\varepsilon := \sum_{k\in\mathbf{Z}_\varepsilon} \varphi^k_\varepsilon(x')1_{I^k_\varepsilon}(x_1)$$

where

$$\varphi^k_\varepsilon(x') = \begin{cases} \varphi(\varepsilon k, x') & \text{if} \quad x' \in I^k_{R_\varepsilon},\ k \in \mathbf{Z}_\varepsilon, \\ 0 & \text{elsewhere.}\end{cases}$$

Let us mention the following straightforward property of $\hat{\varphi}_\varepsilon$:

$$|\varphi - \hat{\varphi}_\varepsilon|_{L^\infty(\mathcal{C}_\varepsilon)} \leq R_\varepsilon|\nabla\varphi|_{L^\infty(\Omega)}. \tag{40}$$

We are in the position to state the main result:

Theorem 2.17 *The limit $u \in W_1$ of (33) verifies (in a weak sense) the following problem:*

$$-\frac{\partial^2 u}{\partial x_1^2} - \left(1 + \frac{\eta}{|D|}\right)\Delta_{x'}u = f \quad in \quad \Omega.$$

Proof. For any $(R_\varepsilon)_\varepsilon \in \mathcal{R}$ and $\varphi \in C^1(\overline{\Omega}) \cap W_1$, let us denote

$$\Phi_\varepsilon = (1 - \frac{r_\varepsilon}{R_\varepsilon} - w_\varepsilon)\varphi + w_\varepsilon \hat{\varphi}_\varepsilon. \tag{41}$$

As a straightforward consequence of the definitions we have $\Phi_\varepsilon \in W_1$ and

$$\lim_{\varepsilon \to 0} |\Phi_\varepsilon - \varphi|_{L^2(\Omega)} = 0. \tag{42}$$

Then, we set in (13) $w = \Phi_\varepsilon$ where Φ_ε is defined by (41).
Then, we get

$$\int_{\Omega \setminus T_\varepsilon} \nabla u_\varepsilon (-\nabla w_\varepsilon)\varphi + \int_{\Omega \setminus T_\varepsilon} \nabla u_\varepsilon \left(1 - \frac{r_\varepsilon}{R_\varepsilon} - w_\varepsilon\right) \nabla \varphi +$$

$$+ \int_{\Omega \setminus T_\varepsilon} \nabla u_\varepsilon \nabla w_\varepsilon \hat{\varphi}_\varepsilon + \int_{\Omega \setminus T_\varepsilon} \nabla u_\varepsilon w_\varepsilon \nabla \hat{\varphi}_\varepsilon + a_\varepsilon \int_{T_\varepsilon} \nabla u_\varepsilon \nabla \hat{\varphi}_\varepsilon =$$

$$= \left(1 - \frac{r_\varepsilon}{R_\varepsilon}\right) \langle f_\varepsilon, \varphi \rangle + \langle f_\varepsilon, w_\varepsilon(\hat{\varphi}_\varepsilon - \varphi) \rangle.$$

Concerning the sum between the first and the third terms of the left-hand side of the previous relation, we have:

$$\left| \int_{\mathcal{C}_\varepsilon} \nabla u_\varepsilon \nabla w_\varepsilon (\hat{\varphi}_\varepsilon - \varphi) \right| \leq \frac{1}{R_\varepsilon} |\hat{\varphi}_\varepsilon - \varphi|_{L^\infty(\mathcal{C}_\varepsilon)} |\nabla u_\varepsilon|_{L^1(\mathcal{C}_\varepsilon)} \leq C |\nabla u_\varepsilon|_{L^2(\mathcal{C}_\varepsilon)} \sqrt{|\mathcal{C}_\varepsilon|} \to 0,$$

where we have used (40).
As a consequence of (24) and (4), the second term converges straightly to

$$\int_\Omega \nabla u \nabla \varphi.$$

The fourth term converges to zero by (20) and (23) as follows:

$$\left| \int_{\Omega \setminus T_\varepsilon} (\nabla u_\varepsilon) w_\varepsilon \nabla \hat{\varphi}_\varepsilon \right| = \left| \int_{\mathcal{C}_\varepsilon} \nabla_{x'} u_\varepsilon (\nabla_{x'} \hat{\varphi}_\varepsilon) w_\varepsilon \right| \leq$$

$$\leq C \int_{\mathcal{C}_\varepsilon} |\nabla_{x'} u_\varepsilon| \leq C |\mathcal{C}_\varepsilon|^{1/2} \to 0.$$

For the fifth one, taking in account (37) and the uniform continuity of $\nabla_{x'}\varphi$, we have

$$a_\varepsilon \int_{T_\varepsilon} \nabla u_\varepsilon \nabla \hat{\varphi}^\varepsilon = a_\varepsilon |T_\varepsilon| \fint_{T_\varepsilon} \nabla_{x'} u_\varepsilon \nabla_{x'} \hat{\varphi}_\varepsilon \to \eta \fint_\Omega \nabla_{x'} u \nabla_{x'} \varphi$$

Then, the left-hand side tends to

$$\int_\Omega \nabla u \nabla \varphi + \eta \fint_\Omega \nabla_{x'} u \nabla_{x'} \varphi.$$

As for the right-hand side, we have $\langle f_\varepsilon, \varphi \rangle \to \langle f, \varphi \rangle$ and

$$|\langle f_\varepsilon, w_\varepsilon(\hat{\varphi}_\varepsilon - \varphi) \rangle| \leq C |\nabla w_\varepsilon|_{\mathcal{C}_\varepsilon} |\hat{\varphi}_\varepsilon - \varphi|_{L^\infty(\mathcal{C}_\varepsilon)} + C |w_\varepsilon|_{L^2(\mathcal{C}_\varepsilon)} |\varphi|_{L^\infty(\mathcal{C}_\varepsilon)}$$

$$\leq C \sqrt{|\mathcal{C}_\varepsilon|} + C |w_\varepsilon|_{L^2(\mathcal{C}_\varepsilon)} \to 0$$

which achieves the proof. ∎

3 The case of separating tubular thin layers

3.1 The conduction problem

We consider the same problem as in the previous section, only this time $\Omega = D \times I$ with $I = (0,1)$ and D is a bounded Lipschitz domain in $\mathbf{R}^2$. Denoting

$$Y := \left(-\frac{1}{2}, +\frac{1}{2}\right)^2, \quad B_1 := B(0,1) = \{y \in \mathbf{R}^2,\ |y| < 1\} \tag{43}$$

$$Y_\varepsilon^k := \varepsilon k + \varepsilon Y, \quad T_\varepsilon^k := B_2(\varepsilon k, \varepsilon d + r_\varepsilon) \setminus \overline{B}_2(\varepsilon k, \varepsilon d - r_\varepsilon), \quad k \in \mathbf{Z}^2, \tag{44}$$

where $B_2(\varepsilon k, r)$ denotes the ball of radius $r > 0$ centered at εk in $\mathbf{R}^2$,

$$\mathbf{Z}_\varepsilon := \{k \in \mathbf{Z}^2, \quad Y_\varepsilon^k \subset D\}. \tag{45}$$

Let $d \in (0, 1/2)$. The distribution of the thin tubes is defined by the following reunion

$$T_\varepsilon = (\cup_{k \in \mathbf{Z}_\varepsilon} T_\varepsilon^k) \times I,$$

where $0 < r_\varepsilon << \varepsilon$. Notice that

$$|T_\varepsilon| \simeq 4\pi d |D| \left(\frac{r_\varepsilon}{\varepsilon}\right) \to 0. \tag{46}$$

We also use the following notation

$$\Omega_\varepsilon = \cup_{k \in \mathbf{Z}_\varepsilon} Y_\varepsilon^k \times I, \quad \partial_D \Omega := \partial D \times (0,1), \quad \partial_3 \Omega := D \times \{0,1\}. \tag{47}$$

In this case, the conduction problem is the following:
To find u_ε solution of

$$-\mathrm{div}(A_\varepsilon \nabla u_\varepsilon) = f_\varepsilon \quad \text{in} \quad \Omega \tag{48}$$

$$u_\varepsilon = 0 \quad \text{on} \quad \partial_3 D \tag{49}$$

$$\frac{\partial u_\varepsilon}{\partial \nu} = 0, \quad \nu = (\nu', 0) \quad \text{on} \quad \partial_D \Omega \tag{50}$$

$$u_\varepsilon = 0 \quad \text{on} \quad \partial \Omega \tag{51}$$

where ν' is the normal on ∂D in the outward direction,

$$A_\varepsilon(x) = \begin{cases} a_\varepsilon(x') & \text{if} \quad x \in \Omega \setminus T_\varepsilon \\ \dfrac{\eta}{|T\varepsilon|} & \text{if} \quad x \in T_\varepsilon \end{cases} \tag{52}$$

where $a_\varepsilon(x') = a\left(\frac{x'}{\varepsilon}\right)$ for some

$$a \in L^\infty_{\rm per}(Y), \tag{53}$$

such that $a(y) \geq a_0 > 0$ for any $y \in Y$.

Let W_2 be the Hilbert space

$$W_2 := \{v \in H^1(\Omega), \quad v = 0 \quad \text{on} \quad \partial_3 D\}$$

endowed with the scalar product

$$(u, v)_{W_2} := (\nabla u, \nabla v)_\Omega. \tag{54}$$

Now, we can present the variational formulation of the problem (48)–(52).

To find $u_\varepsilon \in W_2$ satisfying the following equation

$$\int_{\Omega\setminus T_\varepsilon} a_\varepsilon \nabla u_\varepsilon \nabla v + \eta ⨍_{T_\varepsilon} \nabla u_\varepsilon \nabla v = \langle f_\varepsilon, v\rangle, \quad \forall w \in W_2 \tag{55}$$

where $\langle\cdot,\cdot\rangle$ denotes the duality product between W_2 and W_2'.

Theorem 3.1 *Under the above hypotheses and notations, problem (55) has a unique solution.*

Regarding the relative conductivity outside the fissures, we only assume:

$$a_\varepsilon \geq a_0 > 0, \quad \forall\varepsilon > 0. \tag{56}$$

As for the data, we assume that there exists $f \in W_2'$ such that

$$f_\varepsilon \rightharpoonup f \quad \text{in} \quad W_2'. \tag{57}$$

Proposition 3.2 *We have*

$$(u_\varepsilon)_\varepsilon \quad \textit{is bounded in } W_2. \tag{58}$$

Moreover, there exists $C > 0$, independent of $\varepsilon > 0$, such that

$$⨍_{T_\varepsilon} |\nabla u_\varepsilon|^2 \leq C. \tag{59}$$

Proof. The proof follows the same lines as that of Proposition 2.2. ∎

Proposition 3.3 *Let*

$$p_i^\varepsilon(x) := a\left(\frac{x'}{\varepsilon}\right)\frac{\partial u_\varepsilon}{\partial x_i}(x), \quad i = 1, 2, 3.$$

Then, there exists $p_i \in L^2(\Omega)$ such that, at least on some subsequence:

$$p_i^\varepsilon \rightharpoonup p_i \quad \textit{in} \quad L^2(\Omega). \tag{60}$$

Proof. This is a consequence of (58) and (53).

∎

3.2 Specific tools

For any $(R_\varepsilon)_\varepsilon \in \mathcal{R}$ with $\mathcal{R}$ defined by (18), we define the domain confined between the cylinders of radii $\varepsilon d \pm r_\varepsilon$ and $\varepsilon d \pm R_\varepsilon$ centered at εk, $k \in \mathbf{Z}_\varepsilon$, by:

$$\mathcal{C}^k_{R_\varepsilon} := B^k_2(\varepsilon k, \varepsilon d + R_\varepsilon) \setminus \overline{B}^k_2(\varepsilon k, \varepsilon d + r_\varepsilon)$$

$$\mathcal{C}^k_{-R_\varepsilon} := B^k_2(\varepsilon k, \varepsilon d - R_\varepsilon) \setminus \overline{B}^k_2(\varepsilon k, \varepsilon d - r_\varepsilon)$$

respectively. Then, setting

$$\mathcal{C}^k_\varepsilon := \mathcal{C}^k_{R_\varepsilon} \cup \mathcal{C}^k_{-R_\varepsilon}$$

we define the control zone of our method as the union

$$\mathcal{C}_\varepsilon := \left(\cup_{k\in\mathbf{Z}_\varepsilon} \mathcal{C}^k_\varepsilon\right) \times (0,1).$$

A straightforward computation yields

$$|\mathcal{C}_\varepsilon| = 4\pi d|D| \left(\frac{R_\varepsilon - r_\varepsilon}{\varepsilon}\right) \to 0 \quad \text{as} \quad \varepsilon \to 0. \tag{61}$$

Definition 3.4 *For any $(R_\varepsilon)_\varepsilon \in \mathcal{R}$, we define $w_\varepsilon \in W_2$ by*

$$w_\varepsilon(x) = \begin{cases} 1 & \textit{if} \quad x \in \overline{T}_\varepsilon, \\[2mm] \dfrac{\ln(\varepsilon d + R_\varepsilon) - \ln|x' - \varepsilon k|}{\ln(\varepsilon d + R_\varepsilon) - \ln(\varepsilon d + r_\varepsilon)} & \textit{if} \quad x \in \mathcal{C}^k_{R_\varepsilon} \times I, \\[2mm] \dfrac{\ln|x' - \varepsilon k| - \ln(\varepsilon d - R_\varepsilon)}{\ln(\varepsilon d - r_\varepsilon) - \ln(\varepsilon d - R_\varepsilon)} & \textit{if} \quad x \in \mathcal{C}^k_{-R_\varepsilon} \times I, \\[2mm] 0 & \textit{if} \quad x \in \Omega \setminus (\overline{T}_\varepsilon \cup \mathcal{C}_\varepsilon) \end{cases}$$

Proposition 3.5

$$\begin{gathered} w_\varepsilon \in W_2 \\ 0 \le w_\varepsilon(x) \le 1, \quad \forall x \in \Omega \\ |w_\varepsilon|_\Omega \le |T_\varepsilon \cup \mathcal{C}_\varepsilon|^{1/2} \le C\sqrt{\frac{R_\varepsilon}{\varepsilon}} \to 0. \\ |\nabla w_\varepsilon|^2_{\mathcal{C}_\varepsilon} \simeq \frac{4\pi d|D|}{\varepsilon(R_\varepsilon - r_\varepsilon)} \le \frac{C}{\varepsilon R_\varepsilon} \end{gathered} \tag{62}$$

Definition 3.6 *For any $u \in W_2$ and any $(s_\varepsilon)_{\varepsilon>0}$ with $0 < \varepsilon d + s_\varepsilon < \varepsilon$, we define $G^k_{s_\varepsilon}(u) \in H^{1/2}(I)$ and $G_{s_\varepsilon}(u) \in L^2(\Omega)$ by*

$$G^k_{s_\varepsilon}(u)(x_3) = \frac{1}{2}\left(\fint_{\partial B^k_{s_\varepsilon}} u d\sigma + \fint_{\partial B^k_{-s_\varepsilon}} u d\sigma\right)$$

where $B^k_{\pm s_\varepsilon} = B_2(\varepsilon k, \varepsilon d \pm s_\varepsilon)$,

$$G_{s_\varepsilon}(u)(x', x_3) = \sum_{k \in \mathbf{Z}_\varepsilon} G^k_{s_\varepsilon}(u)(x_3) 1_{Y^k_\varepsilon}(x'), \quad (x', x_3) \in \Omega.$$

Proposition 3.7 *For any* $u \in W_2$*, we have:*

$$⨍_{T_\varepsilon} |G_{s_\varepsilon}(u)|^2 = ⨍_{\Omega_\varepsilon} |G_{s_\varepsilon}(u)|^2. \tag{63}$$

where Ω_ε *is defined by (47).*

Proof. We have

$$⨍_{T_\varepsilon} |G_{s_\varepsilon}(u)|^2 = \frac{|T^0_\varepsilon|}{|T_\varepsilon|} \sum_{k \in \mathbf{Z}_\varepsilon} \int_I |G^k_{s_\varepsilon}(u)|^2 = \frac{1}{|\Omega_\varepsilon|} \sum_{k \in \mathbf{Z}_\varepsilon} \int_{D \times I} (G^k_{s_\varepsilon}(u))^2 1_{Y^k_\varepsilon}$$

as $|T_\varepsilon| = \mathrm{card}\mathbf{Z}_\varepsilon |T^k_\varepsilon|$, $\forall k \in \mathbf{Z}_\varepsilon$, $\varepsilon^2 \mathrm{card}\mathbf{Z}_\varepsilon = |\Omega_\varepsilon|$. ■

Proposition 3.8 *For a.e.* $x_3 \in I$ *and for any* $u \in W_2$*,*

$$\int_{\mathcal{C}^k_{\pm R_\varepsilon}} \nabla u \nabla w_\varepsilon = \frac{\pm 2\pi}{\ln\left(\dfrac{\varepsilon d \pm r_\varepsilon}{\varepsilon d \pm R_\varepsilon}\right)} \left(⨍_{\partial B^k_{\pm R_\varepsilon}} u d\sigma - ⨍_{\partial B^k_{\pm r_\varepsilon}} u d\sigma \right). \tag{64}$$

$$\int_I |G^k_{r_\varepsilon}(u) - G^k_{R_\varepsilon}(u)|^2 (x_3) dx_3 \leq C \frac{R_\varepsilon}{\varepsilon} \int_{\mathcal{C}^k_\varepsilon \times I} |\nabla u|^2 \tag{65}$$

Proof. The identity (64) results from the following direct computation:

$$\int_{\mathcal{C}^k_{-R_\varepsilon}} \nabla u \nabla w_\varepsilon = \int_{\mathcal{C}^k_{-R_\varepsilon}} \mathrm{div}_{x'} (u \nabla_{x'} w_\varepsilon) = \int_{\mathcal{C}^k_{-R_\varepsilon}} u \frac{\partial w_\varepsilon}{\partial \nu'}$$

and likewise in $\mathcal{C}^k_{R_\varepsilon}$.
As for (65), we have:

$$\int_I |G^k_{r_\varepsilon}(u) - G^k_{R_\varepsilon}(u)|^2 (x_3) dx_3 \leq \frac{1}{2} \int_I \left| ⨍_{\partial B^k_{-r_\varepsilon}} u - ⨍_{\partial B^k_{-R_\varepsilon}} u \right|^2 + \frac{1}{2} \int_I \left| ⨍_{\partial B^k_{r_\varepsilon}} u - ⨍_{\partial B^k_{R_\varepsilon}} u \right|^2$$

$$\leq C \left| \ln\left(\frac{\varepsilon d - r_\varepsilon}{\varepsilon d - R_\varepsilon}\right) \right|^2 \int_{\mathcal{C}^k_{-R_\varepsilon} \times I} |\nabla u|^2 \int_{\mathcal{C}^k_{-R_\varepsilon}} |\nabla w_\varepsilon|^2 + C \left| \ln\left(\frac{\varepsilon d + R_\varepsilon}{\varepsilon d + r_\varepsilon}\right) \right|^2 \int_{\mathcal{C}^k_{R_\varepsilon} \times I} |\nabla u|^2 \int_{\mathcal{C}^k_{R_\varepsilon}} |\nabla w_\varepsilon|^2$$

$$\leq C \left| \ln\left(1 + \frac{R_\varepsilon - r_\varepsilon}{\varepsilon d - R_\varepsilon}\right) \right| \int_{\mathcal{C}^k_{-R_\varepsilon} \times I} |\nabla u|^2 + C \left| \ln\left(1 + \frac{R_\varepsilon - r_\varepsilon}{\varepsilon d + r_\varepsilon}\right) \right| \int_{\mathcal{C}^k_{R_\varepsilon} \times I} |\nabla u|^2$$

$$\leq C \frac{R_\varepsilon}{\varepsilon} \int_{\mathcal{C}^k_\varepsilon \times I} |\nabla u|^2$$

■

Proposition 3.9 *For any $u \in W_2$, there holds:*

$$|G_{R_\varepsilon}(u) - G_{r_\varepsilon}(u)|^2_{L^2(\Omega)} \le C\varepsilon R_\varepsilon |\nabla u|^2_{L^2(\mathcal{C}_\varepsilon)} \tag{66}$$

Proof. We have

$$|G_{R_\varepsilon}(u) - G_{r_\varepsilon}(u)|^2_{L^2(\Omega)} = \varepsilon^2 \sum_{k\in \mathbf{Z}_\varepsilon} \int_I |G^k_{r_\varepsilon}(u) - G^k_{R_\varepsilon}(u)|^2(x_3)dx_3 \le$$

$$\le C\varepsilon^2 \sum_{k\in \mathbf{Z}_\varepsilon} \frac{R_\varepsilon}{\varepsilon} \int_{\mathcal{C}^k_\varepsilon \times I} |\nabla u|^2 \le C\varepsilon R_\varepsilon |\nabla u|^2_{L^2(\mathcal{C}_\varepsilon)}.$$

■

Proposition 3.10 *For any $u \in W_2$, there holds:*

$$|u - G_{R_\varepsilon}(u)|_{L^2(\Omega_\varepsilon)} \le C\varepsilon |\nabla u|_{L^2(\Omega)}. \tag{67}$$

Proof. We have:

$$\sum_{k\in \mathbf{Z}_\varepsilon} \int_{Y^k_\varepsilon \times I} |u - G^k_{R_\varepsilon}(u)|^2 \le \sum_{k\in \mathbf{Z}_\varepsilon} \left(\int_{B_2(\varepsilon k, \varepsilon/\sqrt{2})} |u - \fint_{\partial B^k_{-R_\varepsilon}} u|^2 + \int_{B_2(\varepsilon k, \varepsilon/\sqrt{2})} |u - \fint_{\partial B^k_{R_\varepsilon}} u|^2 \right)$$

$$\le C\varepsilon^2 \sum_{k\in \mathbf{Z}_\varepsilon} \int_{B_2(\varepsilon k, \varepsilon/\sqrt{2}) \times I} |\nabla u|^2 \le C\varepsilon^2 |\nabla u|^2_{L^2(\Omega)}$$

where $B_2(\varepsilon k, \varepsilon/\sqrt{2})$ denotes the ball of $\mathbf{R}^2$ of radius $\dfrac{\varepsilon}{\sqrt{2}}$ centered at εk, $k \in \mathbf{Z}_\varepsilon$.

■

A straightforward computation shows that the first eigenvalue of the Neumann problem in T^k_ε is of r_ε^{-2} order. Then, the following variant of Poincaré-Wirtinger inequality holds true:

$$|u - G^k_0(u)|_{L^2(T^k_\varepsilon)} \le C r_\varepsilon |\nabla u|_{L^2(T^k_\varepsilon)}, \quad \forall u \in H^1(T^k_\varepsilon), \quad k \in \mathbf{Z}_\varepsilon. \tag{68}$$

Proposition 3.11 *For any $u \in W_2$ and for any $k \in \mathbf{Z}_\varepsilon$,*

$$\int_{T^k_\varepsilon \times I} |u - G^k_{r_\varepsilon}(u)|^2 \le C\varepsilon r_\varepsilon |\nabla_{x'} u|^2_{T^k_\varepsilon \times I} \tag{69}$$

Proof. Let $k \in \mathbf{Z}_\varepsilon$. We have:

$$\int_{T^k_\varepsilon \times I} |u - G^k_{r_\varepsilon}(u)|^2 \le 2 \int_I \left(\int_{T^k_\varepsilon} |u - G^k_0(u)|^2 + \int_{T^k_\varepsilon} |G^k_0(u) - G^k_{r_\varepsilon}(u)|^2 \right).$$

The first integral of the right-hand side is estimated through (68) and for the second, we use the same argument as in (66). ■

Proposition 3.12 *For any $u \in W_2$, there holds*

$$|u - G_{r_\varepsilon}(u)|^2_{L^2(T_\varepsilon)} \le C\varepsilon r_\varepsilon |\nabla_{x'} u|^2_{L^2(T_\varepsilon)}. \tag{70}$$

Proof. We have

$$|u - G_{r_\varepsilon}(u)|^2_{L^2(T_\varepsilon)} \le \sum_{k\in\mathbf{Z}_\varepsilon} \int_{T^k_\varepsilon\times I} |u - G^k_{r_\varepsilon}(u)|^2 \le \sum_{k\in\mathbf{Z}_\varepsilon} C\varepsilon r_\varepsilon |\nabla_{x'} u|^2_{T^k_\varepsilon\times I}$$

■

Proposition 3.13 *For any $u \in W_2$, there holds*

$$⨍_{T_\varepsilon} |u|^2 \le C|\nabla u|^2_{L^2(\Omega)} \tag{71}$$

Proof. We have:

$$\begin{aligned}
⨍_{T_\varepsilon} |u|^2 &\le 2⨍_{T_\varepsilon} |u - G_{r_\varepsilon}(u)|^2 + 2⨍_{T_\varepsilon} |G_{r_\varepsilon}(u)|^2 \\
&\le 2⨍_{T_\varepsilon} |u - G_{r_\varepsilon}(u)|^2 + 2⨍_{\Omega_\varepsilon} |G_{r_\varepsilon}(u)|^2 \\
&\le C\varepsilon r_\varepsilon ⨍_{T_\varepsilon} |\nabla u|^2 + 2⨍_{\Omega_\varepsilon} |G_{r_\varepsilon}(u)|^2 \\
&\le C\left(\frac{\varepsilon r_\varepsilon}{|T_\varepsilon|}\int_{T_\varepsilon}|\nabla u|^2 + |G_{r_\varepsilon}(u) - G_{R_\varepsilon}(u)|^2_{L^2(\Omega_\varepsilon)} + |G_{R_\varepsilon}(u) - u|^2_{L^2(\Omega_\varepsilon)} + |u|^2_{L^2(\Omega)}\right) \\
&\le C|\nabla u|^2_{L^2(\Omega)}
\end{aligned}$$

where we have used (66), (67) and (70). ■

Corollary 3.14 *The following estimate holds true:*

$$⨍_{T_\varepsilon} |u_\varepsilon|^2 \le C \tag{72}$$

for some constant $C > 0$ independent of $\varepsilon > 0$.

Definition 3.15 *For every $\varphi \in C(\overline{\Omega}) \cap W_2$, define*

$$M_{r_\varepsilon}(\varphi) = \sum_{k\in\mathbf{Z}_\varepsilon} \left(⨍_{Y^k_\varepsilon} \varphi(y, x_3)dy\right) 1_{T^k_\varepsilon}(x')$$

Proposition 3.16 *There holds*

$$\lim_{\varepsilon\to 0} ⨍_{T_\varepsilon} |\varphi - M_{r_\varepsilon}(\varphi)|^2 = 0, \quad \forall\varphi \in C(\overline{\Omega}) \cap W_2. \tag{73}$$

Proof. Let $\varphi \in C(\overline{\Omega}) \cap W_2$ and let $\delta > 0$. The uniform continuity of φ yields the existence of some $\varepsilon_0 > 0$ such that:

$$\forall k \in \mathbf{Z}_\varepsilon,\ \forall x_3 \in I,\ \int_{T^k_\varepsilon} |\varphi(x', x_3) - ⨍_{Y^k_\varepsilon} \varphi(y, x_3)dy|^2 dx' \le \delta,\ \forall\varepsilon \in (0, \varepsilon_0).$$

There results

$$\fint_{T_\varepsilon} |\varphi - M_{r_\varepsilon}(\varphi)|^2 \leq \delta, \quad \forall \varepsilon \in (0, \varepsilon_0).$$

■

Proposition 3.17 *There holds*

$$\fint_{T_\varepsilon} G_{r_\varepsilon}(u_\varepsilon) M_{r_\varepsilon}(\varphi) = \fint_{\Omega_\varepsilon} G_{r_\varepsilon}(u_\varepsilon)\varphi, \quad \forall \varphi \in C(\overline{\Omega}) \cap W_2. \tag{74}$$

Proof. Let $\varphi \in C(\overline{\Omega}) \cap W_2$. We have:

$$\fint_{T_\varepsilon} G_{r_\varepsilon}(u_\varepsilon) M_{r_\varepsilon}(\varphi) = \frac{|T_\varepsilon^0|}{|T_\varepsilon|} \sum_{k \in \mathbf{Z}_\varepsilon} \int_I G_{r_\varepsilon}^k(u)(x_3) \left(\fint_{Y_\varepsilon^k} \varphi \right) = \frac{|T_\varepsilon^0|}{\varepsilon^2 |T_\varepsilon|} \int_{\Omega_\varepsilon} G_{r_\varepsilon}(u)\varphi.$$

■

3.3 The homogenization result

A preliminary result is the following:

Proposition 3.18 *There exists $u \in W_2$ such that, on some subsequence,*

$$u_\varepsilon \rightharpoonup u \quad \textit{in} \quad H^1(\Omega). \tag{75}$$

$$G_{R_\varepsilon}(u_\varepsilon) \to u \quad \textit{in} \quad L^2(\Omega) \tag{76}$$

$$G_{r_\varepsilon}(u_\varepsilon) \to u \quad \textit{in} \quad L^2(\Omega) \tag{77}$$

Proof. From (58), we get, on some subsequence, the convergence (75). The convergence (76) follows from (75) and (67). Finally, the convergence (77) is a consequence of (76) and (66). ■

Proposition 3.19 *For any $\varphi \in W_2$, the following convergences hold true on some subsequence:*

$$\fint_{T_\varepsilon} u_\varepsilon \varphi \to \fint_{\Omega} u\varphi, \tag{78}$$

$$\fint_{T_\varepsilon} \frac{\partial u_\varepsilon}{\partial x_3}\varphi \to \fint_{\Omega} \frac{\partial u}{\partial x_3}\varphi, \tag{79}$$

where u was introduced by (75).

Proof. Let $\varphi \in C(\Omega) \cap W_2$. We have

$$\fint_{T_\varepsilon} u_\varepsilon \varphi = \fint_{T_\varepsilon} (u_\varepsilon - G_{r_\varepsilon}(u_\varepsilon))\varphi + \fint_{T_\varepsilon} G_{r_\varepsilon}(u_\varepsilon)(\varphi - M_{r_\varepsilon}(\varphi)) + \fint_{T_\varepsilon} G_{r_\varepsilon}(u_\varepsilon) M_{r_\varepsilon}(\varphi)$$

where, taking into account (70), (73), (75) and (77), we have:

$$\left|\fint_{T_\varepsilon}(u_\varepsilon - G_{r_\varepsilon}(u_\varepsilon))\varphi\right| \le \left(\fint_{T_\varepsilon}|u - G_{r_\varepsilon}(u)|^2\right)^{1/2}\left(\fint_{T_\varepsilon}|\varphi|^2\right)^{1/2}$$

$$\le (\varepsilon r_\varepsilon)^{1/2}|\varphi|_{L^\infty(\Omega)}\left(\fint_{T_\varepsilon}|\nabla u_\varepsilon|^2\right)^{1/2} \le C\varepsilon \to 0$$

$$\left|\fint_{T_\varepsilon} G_{r_\varepsilon}(u_\varepsilon)(\varphi - M_{r_\varepsilon}(\varphi))\right| \le |G_{r_\varepsilon}(u_\varepsilon)|_\Omega \left(\fint_{T_\varepsilon}|\varphi - M_{r_\varepsilon}(\varphi)|^2\right)^{1/2} \to 0.$$

Then, using (74), we get

$$\lim_{\varepsilon\to 0}\fint_{T_\varepsilon} u_\varepsilon\varphi = \lim_{\varepsilon\to 0}\fint_{\Omega_\varepsilon} G_{r_\varepsilon}(u_\varepsilon)\varphi = \fint_\Omega u\varphi.$$

To conclude, we notice that $C(\Omega)\cap W_2$ is dense in W_2 and that the following estimate yields the continuity of $\varphi \mapsto \fint_{T_\varepsilon} u_\varepsilon\varphi$ as a mapping:

$$\left|\fint_{T_\varepsilon} u_\varepsilon\varphi\right| \le \left(\fint_{T_\varepsilon}|u_\varepsilon|^2\right)^{1/2}\left(\fint_{T_\varepsilon}|\varphi|^2\right)^{1/2} \le C|\nabla\varphi|_\Omega.$$

For $\varphi \in \mathcal{D}(\Omega)$, we have

$$\int_{T_\varepsilon}\frac{\partial u_\varepsilon}{\partial x_3}\varphi = -\int_{T_\varepsilon} u_\varepsilon\frac{\partial\varphi}{\partial x_3} + \int_{\partial T_\varepsilon} u_\varepsilon\varphi\nu_3 = -\int_{T_\varepsilon} u_\varepsilon\frac{\partial\varphi}{\partial x_3} + \sum_{k\in\mathbf{Z}_\varepsilon}\int_{\partial\mathcal{C}^k_{R_\varepsilon}\times(0,1)} u_\varepsilon\varphi\nu_3 =$$

As $\nu_3 = 0$ on $\partial\mathcal{C}^k_{R_\varepsilon}\times(0,1)$, $k\in\mathbf{Z}_\varepsilon$, we infer that

$$\fint_{T_\varepsilon}\frac{\partial u_\varepsilon}{\partial x_3}\varphi + \fint_{T_\varepsilon} u_\varepsilon\frac{\partial\varphi}{\partial x_3} = 0.$$

There results:

$$\left|\fint_{T_\varepsilon}\frac{\partial u_\varepsilon}{\partial x_3}\varphi\right| \le \sqrt{\fint_{T_\varepsilon}|u_\varepsilon|^2}|\nabla\varphi|_\Omega, \quad \forall\varphi\in\mathcal{D}(\Omega)$$

and (72) yields the existence of some $v_3 \in H^{-1}(\Omega)$ such that

$$\frac{1}{|T_\varepsilon|}1_{T_\varepsilon}\frac{\partial u_\varepsilon}{\partial x_3} \rightharpoonup v_3 \quad \text{in} \quad H^{-1}(\Omega).$$

Then, passing to the limit as $\varepsilon \to 0$, we get

$$\langle v_3, \varphi\rangle + \fint_\Omega u\frac{\partial\varphi}{\partial x_3} = 0, \tag{80}$$

that is, $v_3 = \dfrac{1}{|\Omega|}\dfrac{\partial u}{\partial x_3} \in L^2(\Omega)$. ■

Definition 3.20 *For each $i \in \{1,2\}$, let w^i denote the solution of:*

$$-\mathrm{div}_y(a(y)\nabla_y w^i) = \frac{\partial a}{\partial y_i} \quad \textit{in} \quad Y$$

$$w^i \in H^1_{\mathrm{per}}(Y), \quad \fint_Y w^i = 0.$$

Now, we are in the position to state our main result:

Theorem 3.21 *The limit $u \in W_2$ of (75) verifies (in a weak sense) the following problem:*

$$-\mathrm{div}_{x'}(A^{\mathrm{hom}} \nabla_{x'} u) - \left(\fint_Y a(y)dy + \frac{\eta}{|D|} \right) \frac{\partial^2 u}{\partial x_3^2} = f \quad \textit{in} \quad \Omega, \tag{81}$$

where the homogenized matrix $A^{\mathrm{hom}} \in \mathbf{R}^{2\times 2}$ is defined by its coefficients:

$$a^{\mathrm{hom}}_{ij} = \fint_Y a(y) \left(\delta_{ij} + \frac{\partial w^i}{\partial y_j} \right) dy, \quad i,j \in \{1,2\}. \tag{82}$$

Proof. For any $(R_\varepsilon)_\varepsilon \in \mathcal{R}$ and $\varphi \in C^1(\overline{\Omega}) \cap W_2$, let us denote

$$\Phi_\varepsilon = (1 - w_\varepsilon)\varphi + w_\varepsilon G_0(\varphi). \tag{83}$$

As a straightforward consequence of the definition, we have $\Phi_\varepsilon \in W_2$ and

$$\lim_{\varepsilon \to 0} |\Phi_\varepsilon - \varphi|_\Omega = 0 \tag{84}$$

Then, we set in (55) $w = \Phi_\varepsilon$ where Φ_ε is defined by (83).
Then, we get

$$\int_{\Omega \setminus T_\varepsilon} a_\varepsilon \nabla u_\varepsilon (-\nabla w_\varepsilon)\varphi + \int_{\Omega \setminus T_\varepsilon} a_\varepsilon \nabla u_\varepsilon \,(1 - w_\varepsilon)\, \nabla \varphi +$$
$$+ \int_{\Omega \setminus T_\varepsilon} a_\varepsilon \nabla u_\varepsilon \nabla w_\varepsilon G_0(\varphi) + \int_{\Omega \setminus T_\varepsilon} a_\varepsilon \nabla u_\varepsilon w_\varepsilon \nabla G_0(\varphi) + \eta \fint_{T_\varepsilon} \nabla u_\varepsilon \nabla G_0(\varphi) =$$
$$= \langle f_\varepsilon, \varphi \rangle + \langle f_\varepsilon, w_\varepsilon (G_0(\varphi) - \varphi) \rangle.$$

Concerning the sum between the first and the third terms of the left-hand side of the previous relation, we have:

$$\left| \int_{\mathcal{C}_\varepsilon} a_\varepsilon \nabla u_\varepsilon \nabla w_\varepsilon (G_0(\varphi) - \varphi) \right| \le \frac{1}{\sqrt{\varepsilon R_\varepsilon}} |G_0(\varphi) - \varphi|_{L^\infty(\mathcal{C}_\varepsilon)} |\nabla u_\varepsilon|_{L^2(\mathcal{C}_\varepsilon)} \le C \sqrt{\frac{R_\varepsilon}{\varepsilon}} \to 0.$$

As for the second term, we have:

$$\int_{\Omega \setminus T_\varepsilon} a_\varepsilon \nabla u_\varepsilon \,(1 - w_\varepsilon)\, \nabla \varphi = \int_{\Omega \setminus \mathcal{C}_\varepsilon} a_\varepsilon \nabla u_\varepsilon \nabla \varphi + \int_{\mathcal{C}_\varepsilon \setminus T_\varepsilon} a_\varepsilon \nabla u_\varepsilon \,(1 - w_\varepsilon)\, \nabla \varphi$$

with

$$\left| \int_{\mathcal{C}_\varepsilon \setminus T_\varepsilon} a_\varepsilon \nabla u_\varepsilon \,(1 - w_\varepsilon)\, \nabla \varphi \right| \le C |\nabla u_\varepsilon|_{L^2(\Omega)} |\nabla \varphi|_{L^2(\mathcal{C}_\varepsilon)} \le C |\nabla \varphi|_{L^2(\mathcal{C}_\varepsilon)} \to 0.$$

Moreover:

$$\int_{\Omega \setminus \mathcal{C}_\varepsilon} a_\varepsilon \nabla u_\varepsilon \nabla \varphi = \int_\Omega p^\varepsilon \nabla \varphi 1_{\Omega \setminus \mathcal{C}_\varepsilon} \to \int_\Omega p \nabla \varphi.$$

where p is defined by (60). Next, we shall identify p. For this, we denote $w^i_\varepsilon(x') := w^i\left(\dfrac{x'}{\varepsilon}\right)$, the periodization of w^i. For $\psi \in \mathcal{D}(\Omega)$ and $i = 1,2$, we have:

$$\int_\Omega p^\varepsilon_i \psi = \int_\Omega a_\varepsilon \frac{\partial u_\varepsilon}{\partial x_i}\psi = \sum_{j=1,2}\left(\int_\Omega \left(a_\varepsilon \delta_{ij} + \varepsilon a_\varepsilon \frac{\partial w^i_\varepsilon}{\partial x_j}\right)\frac{\partial u_\varepsilon}{\partial x_j}\psi - \varepsilon \int_\Omega a_\varepsilon \frac{\partial w^i_\varepsilon}{\partial x_j}\frac{\partial u_\varepsilon}{\partial x_j}\psi\right)$$

The second term converges to zero because:

$$\left|\varepsilon \int_\Omega a_\varepsilon \frac{\partial w^i_\varepsilon}{\partial x_j}\frac{\partial u_\varepsilon}{\partial x_j}\psi\right| \le C\sqrt{\varepsilon}|\nabla_{x'} u_\varepsilon|_{L^2(\Omega)} \le C\sqrt{\varepsilon}.$$

Noticing that

$$\sum_{j=1}^{3}\frac{\partial}{\partial x_j}\left(a_\varepsilon \delta_{ij} + \varepsilon a_\varepsilon \frac{\partial w^i_\varepsilon}{\partial x_j}\right) = 0 \quad \text{in} \quad \mathcal{D}'(D), \quad i = 1,2,$$

and using the Curl-Div lemma (see [7], [8]) we obtain by passing to the limit:

$$\int_\Omega p_i \psi = \int_\Omega a^{\text{hom}}_{ij}\frac{\partial u}{\partial x_j}\psi,$$

that is,

$$p_i = \sum_{j=1,2} a^{\text{hom}}_{ij}\frac{\partial u}{\partial x_j}, \quad i = 1,2.$$

The identification of p_3 follows from

$$p^\varepsilon_3 = a_\varepsilon \frac{\partial u_\varepsilon}{\partial x_3} \rightharpoonup \left(⨍_Y a(y)dy\right)\frac{\partial u}{\partial x_3} \quad \text{in} \quad L^2(\Omega).$$

The fourth term converges to zero by (62) and (61) as follows:

$$\left|\int_{\Omega\setminus T_\varepsilon} \nabla u_\varepsilon w_\varepsilon \nabla G_0(\varphi)\right| \le \sum_{k\in \mathbf{Z}_\varepsilon}\left|\int_{Y^k_\varepsilon \setminus T^k_\varepsilon}\frac{\partial u_\varepsilon}{\partial x_3} w_\varepsilon \frac{\partial G^k_0(\varphi)}{\partial x_3}\right| \le$$

$$\le \sum_{k\in \mathbf{Z}_\varepsilon}\int_I\left|\int_{\mathcal{C}^k_{R_\varepsilon}\setminus T^k_\varepsilon}\frac{\partial u_\varepsilon}{\partial x_3} w_\varepsilon G^k_0\left(\frac{\partial \varphi}{\partial x_3}\right)\right| \le |\nabla u_\varepsilon|_{L^2(\Omega)}\left|\frac{\partial\varphi}{\partial x_3}\right|_{L^\infty(\Omega)}\sqrt{|\mathcal{C}_\varepsilon|} \to 0$$

For the fifth one, taking in account (79) and the uniform continuity of $\dfrac{\partial \varphi}{\partial x_3}$, we have

$$\eta ⨍_{T_\varepsilon} \nabla u \nabla G_0(\varphi) = \eta ⨍_{T_\varepsilon}\frac{\partial u_\varepsilon}{\partial x_3}\frac{\partial G_0(\varphi)}{\partial x_3} = \eta ⨍_{T_\varepsilon}\frac{\partial u_\varepsilon}{\partial x_3} G_0\left(\frac{\partial\varphi}{\partial x_3}\right) \to \eta ⨍_\Omega \frac{\partial u}{\partial x_3}\frac{\partial \varphi}{\partial x_3}.$$

Then, the left-hand side tends to the left-hand side of (81).
As for the right-hand side, we have $\langle f_\varepsilon, \varphi\rangle \to \langle f, \varphi\rangle$ and

$$|\langle f_\varepsilon, w_\varepsilon(G_0(\varphi)-\varphi)\rangle| \le C|\nabla w_\varepsilon|_{\mathcal{C}_\varepsilon}|G_0(\varphi)-\varphi|_{L^\infty(\mathcal{C}_\varepsilon)} + C|w_\varepsilon|_{L^2(\mathcal{C}_\varepsilon\cup\overline{T}_\varepsilon)}|\varphi|_{L^2(\mathcal{C}_\varepsilon\cup\overline{T}_\varepsilon)}$$

$$\le \sqrt{\frac{R_\varepsilon}{\varepsilon}}\,|\varphi|_\infty \to 0,$$

which achieves the proof. ■

Remark 3.22 *The conduction problem in the presence of ε-periodic highly conductive thin layers confined between spheres of ε-order radii can be homogenized similarly. In that case the macroscopic system does not present any increase of the conductivity. Therefore, we conjecture that the presence of ε-periodic highly conductive thin layers determines an increase of the macroscopic conductivity in a certain direction only if these layers cover entirely segment lines of unity order having this direction.*

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References

[1] M. Bellieud, G. Bouchitté. Homogenization of elliptic problems in a fiber reinforced structure. Non local effects. *Ann. Scuola Norm. Sup. Pis Cl. Sci.(4)* **26 (3)** (1998) 407–436.

[2] F. Bentalha, I. Gruais, D. Polisevski, Homogenization of a conductive suspension in a Stokes-Boussinesq flow, *Applicable Analysis.***85 (6/7)**, (2006) 811–830.

[3] F. Bentalha, I. Gruais, D. Polisevski, Diffusion process in a rarefied binary structure, *Revue Roumaine de Mathématiques Pures et Appliquées.* **52(1)**, (2007) 9–34.

[4] F. Bentalha, I. Gruais, D. Polisevski, Diffusion in a highly rarefied binary structure of general periodic shape, to appear in *Applicable Analysis*, (2008).

[5] G. Bouchitté, C. Picard Singular perturbations and homogenization in stratified media. *Applicable Analysis.* **61(3/4)**, (1996) 307–341.

[6] B. Héron, J. Mossino, C. Picard Homogenization of some quasilinear problems for stratified media with low and high conductivities *Differential Integral Equations* **7(1)** (1994) 157–178.

[7] F. Murat. Compacité par Compensation. *Ann. Scuola Norm. Sup. Pisa Cl. Sci.* **5(4)** (1978)

[8] F. Murat. Compacité par Compensation II, in *Recent Methods in Nonlinear Analysis*, E. De Giorgi, E. Magenes, U. Mosco Edit rice, (Bologna) 1979.

[9] F. Murat, L. Tartar. Calcul des variations et homogénéisation In *Homogenization methods: theory and applications in physics (Bréau-sans-Nappe, 1983)*, Collect. Dir. Études Rech. Élec. France **57**, Eyrolles, Paris Paris, Eyrolles, 1985, 319–370.

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